Phys. Rev. Lett. (2007), in print

Refuting the odd number limitation of time-delayed feedback control

B. Fiedler,[1] V. Flunkert,[2] M. Georgi,[1] P. Hövel,[2] and E. Schöll[2]

[1]*Institut für Mathematik I, FU Berlin, Arnimallee 2-6, D-14195 Berlin, Germany*

[2]*Institut für Theoretische Physik, Technische Universität Berlin, Hardenbergstraße 36, 10623 Berlin, Germany*

(Dated: February 3, 2008)

We refute an often invoked theorem which claims that a periodic orbit with an odd number of real Floquet multipliers greater than unity can never be stabilized by time-delayed feedback control in the form proposed by Pyragas. Using a generic normal form, we demonstrate that the unstable periodic orbit generated by a subcritical Hopf bifurcation, which has a single real unstable Floquet multiplier, can in fact be stabilized. We derive explicit analytical conditions for the control matrix in terms of the amplitude and the phase of the feedback control gain, and present a numerical example. Our results are of relevance for a wide range of systems in physics, chemistry, technology, and life sciences, where subcritical Hopf bifurcations occur.

PACS numbers: 05.45.Gg, 02.30.Ks
Keywords: control, time-delayed feedback

The stabilization of unstable and chaotic systems is a central issue in applied nonlinear science [1, 2, 3]. Starting with the work of Ott, Grebogi and Yorke [4], a variety of methods have been developed in order to stabilize unstable periodic orbits (UPOs) embedded in a chaotic attractor by employing tiny control forces. A particularly simple and efficient scheme is time-delayed feedback as suggested by Pyragas [5]. It is an attempt to stabilize periodic orbits of minimal period T by a feedback control which involves a time delay $\tau = nT$, for suitable positive integer n. A linear feedback example is

$$\dot{z}(t) = f(\lambda, z(t)) + B[z(t-\tau) - z(t)] \tag{1}$$

where $\dot{z}(t) = f(\lambda, z(t))$ describes a d-dimensional nonlinear dynamical system with bifurcation parameter λ and an unstable orbit of period T. B is a suitably chosen constant feedback control matrix. Typical choices are multiples of the identity or of rotations, or matrices of low rank. More general nonlinear feedbacks are conceivable, of course. The main point, however, is that the Pyragas choice $\tau_P = nT$ of the delay time eliminates the feedback term in case of successful stabilization and thus recovers the original T-periodic solution $z(t)$. In this sense the method is noninvasive. Although time delayed feedback control has been widely used with great success in real world problems in physics, chemistry, biology, and medicine, e.g. [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18], severe limitations are imposed by the common belief that certain orbits cannot be stabilized for any strength of the control force. In fact, it has been contended that periodic orbits with an odd number of real Floquet multipliers greater than unity cannot be stabilized by the Pyragas method [19, 20, 21, 22, 23, 24], even if the simple scheme (1) is extended by multiple delays in form of an infinite series [25]. To circumvent this restriction other, more complicated, control schemes, like an oscillating feedback [26], or the introduction of an additional, unstable degree of freedom [24, 27], have been proposed. In this letter, we claim, and show by example, that the general limitation for orbits with an odd number of real unstable Floquet multipliers greater than unity does not hold, but that stabilization may be possible for suitable choices of B. We illustrate this with an example which consists of an unstable periodic orbit generated by a subcritical Hopf bifurcation, refuting the theorem in [20].

Consider the normal form of a subcritical Hopf bifurcation, extended by a time delayed feedback term

$$\dot{z}(t) = \left[\lambda + i + (1+i\gamma)|z(t)|^2\right] z(t) + b[z(t-\tau) - z(t)] \tag{2}$$

with $z \in \mathbb{C}$ and real parameters λ and γ. Here the Hopf frequency is normalized to unity. The feedback matrix B is represented by multiplication with a complex number $b = b_R + ib_I = b_0 e^{i\beta}$ with real b_R, b_I, β, and positive b_0. Note that the nonlinearity $f(\lambda, z(t)) = \left[\lambda + i + (1+i\gamma)|z(t)|^2\right] z(t)$ commutes with complex rotations. Hence the Hopf bifurcations from the trivial solution $z \equiv 0$ at simple imaginary eigenvalue $\eta = i\omega \neq 0$ produce rotating wave solutions $z(t) = z(0)\exp\left(i\frac{2\pi}{T}t\right)$ with period T even in the nonlinear case and with delay terms. This follows from uniqueness of the emanating Hopf branches.

Transforming Eq. (2) to amplitude and phase variables r, θ using $z(t) = r(t)e^{i\theta(t)}$, we obtain at $b = 0$

$$\dot{r}(t) = \left(\lambda + r^2\right) r \tag{3}$$

$$\dot{\theta}(t) = 1 + \gamma r^2. \tag{4}$$

An unstable periodic orbit (UPO) with $r^2 = -\lambda$ and period $T = 2\pi/(1-\gamma\lambda)$ exists for $\lambda < 0$. At $\lambda = 0$ a subcritical Hopf bifurcation occurs. The Pyragas control method chooses delays as $\tau_P = nT$. This defines the local *Pyragas curve* in the (λ, τ)-plane for any $n \in \mathbb{N}$

$$\tau_P(\lambda) = \frac{2\pi n}{1-\gamma\lambda} = 2\pi n(1 + \gamma\lambda + \dots) \tag{5}$$

which emanates from the Hopf bifurcation point $\lambda = 0$. Under further nondegeneracy conditions, the Hopf point

$\lambda = 0$, $\tau = nT$ ($n \in \mathbb{N}_0$) continues to a Hopf bifurcation curve $\tau_H(\lambda)$ for $\lambda < 0$. We determine this *Hopf curve* next. It is characterized by purely imaginary eigenvalues $\eta = i\omega$ of the transcendental characteristic equation

$$\eta = \lambda + i + b\left(e^{-\eta\tau} - 1\right) \tag{6}$$

which results from the linearization at the steady state $z = 0$ of the delayed system (2).

Separating Eq. (6) into real and imaginary parts

$$0 = \lambda + b_0[\cos(\beta - \omega\tau) - \cos\beta] \tag{7}$$

$$\omega - 1 = b_0[\sin(\beta - \omega\tau) - \sin\beta] \tag{8}$$

and using trigonometric identities to eliminate $\omega(\lambda)$ yields an explicit expression for the multivalued Hopf curve $\tau_H(\lambda)$ for given control amplitude b_0 and phase β:

$$\tau_H = \frac{\pm\arccos\left(\frac{b_0\cos\beta - \lambda}{b_0}\right) + \beta + 2\pi n}{1 - b_0\sin\beta \pm \sqrt{\lambda(2b_0\cos\beta - \lambda) + b_0^2\sin^2\beta}}. \tag{9}$$

Note that τ_H is not defined in case of $\beta = 0$ and $\lambda < 0$. Thus complex b is a necessary condition for the existence of the Hopf curve in the subcritical regime $\lambda < 0$. Fig. 1 displays the family of Hopf curves, $n \in \mathbb{N}_0$, Eq. (9), and the Pyragas curve $n = 1$, Eq. (5), in the (λ, τ) plane. In Fig. 1(b) the domains of instability of the trivial steady state $z = 0$, bounded by the Hopf curves, are marked by light grey shading (yellow online). The dimensions of the unstable manifold of $z = 0$ are given in parentheses along the τ-axis in Fig. 1(b). By construction, the period of the bifurcating periodic orbits becomes equal to $\tau_P = nT$ along the Pyragas curve, since the time-delayed feedback term vanishes. Standard exchange of

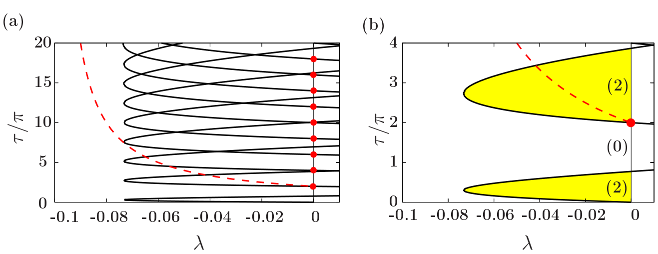

FIG. 1: (Color online) Pyragas (red dashed) and Hopf (black solid) curves in the (λ, τ)-plane: (a) Hopf bifurcation curves $n = 0, ..., 10$, (b) Hopf bifurcation curves $n = 0, 1$ in an enlarged scale. Yellow shading marks the domains of unstable $z = 0$ and numbers in parentheses denote the dimension of the unstable manifold of $z = 0$ ($\gamma = -10$, $b_0 = 0.3$ and $\beta = \pi/4$).

stability results [28], which hold verbatim for delay equations, then assert that the bifurcating branch of periodic solutions locally inherits linear asymptotic (in)stability from the trivial steady state, i.e., it consists of stable periodic orbits on the Pyragas curve $\tau_P(\lambda)$ inside the yellow domains for small $|\lambda|$. Note that an unstable trivial steady state is not a sufficient condition for stabilization of the subcritical orbit, but other (e.g., global) bifurcations at $\lambda < 0$ must be considered as well. More precisely, for small $|\lambda|$ the unstable periodic orbits possess a single Floquet multiplier $\mu = \exp(\Lambda\tau) \in (1, \infty)$, near unity, which is simple. All other nontrivial Floquet multipliers lie strictly inside the complex unit circle. In particular, the (strong) unstable dimension of these periodic orbits is odd, here 1, and their unstable manifold is two-dimensional. This is shown in Fig. 2, which depicts solutions Λ of the characteristic equation of the periodic solution on the Pyragas curve. Panel (a) (top) shows the

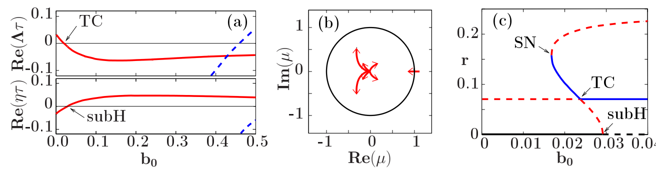

FIG. 2: (Color online) (a) top: Real part of Floquet exponents Λ of the periodic orbit vs. feedback amplitude b_0. bottom: Real part of eigenvalue η of steady state vs. feedback amplitude b_0. (b): Floquet multipliers $\mu = \exp(\Lambda\tau)$ (red) in the complex plane with the feedback amplitude $b_0 \in [0, 0.3]$ as a parameter. (c): radii of periodic orbits. Solid (dashed) lines correspond to stable (unstable) orbits. ($\lambda = -0.005$, $\gamma = -10$, $\tau = \frac{2\pi}{1-\gamma\lambda}$, $\beta = \pi/4$).

dependence of the real part of the critical Floquet exponent Λ on the amplitude of the feedback gain b_0. The largest real part is positive for $b_0 = 0$. Thus the periodic orbit is unstable. As the amplitude of the feedback gain increases, the largest real part of the eigenvalue becomes smaller and eventually changes sign. Hence the periodic orbit is stabilized. Note that an infinite number of Floquet exponents are created by the control scheme; their real parts tend to $-\infty$ in the limit $b_0 \to 0$, and some of them may cross over to positive real parts for larger b_0 (blue curve), terminating the stability of the periodic orbit. Panel (b) of Fig. 2 shows the behavior of the Floquet multipliers $\mu = \exp(\Lambda\tau)$ in the complex plane with the increasing amplitude of the feedback gain b_0 as a parameter (marked by arrows). There is an isolated real multiplier crossing the unit circle at $\mu = 1$, in contrast to the result stated in [20]. This is caused by a transcritical bifurcation (TC) in which the subcritical Pyragas orbit (whose radius is given by $r = (-\lambda)^{1/2}$ independently of the control amplitude b_0) collides with a delay-induced periodic orbit, as shown in Fig. 2(c). This delay-induced orbit is generated at a finite value of the control amplitude b_0 (SN) by a saddle-node bifurcation (collision with another unstable delay-induced periodic orbit). At TC, the subcritical orbit and the delay-induced orbit exchange stability. The latter vanishes at a subcritical Hopf (subH) bifurcation at which the trivial steady state becomes unstable. Except at TC, the delay-induced orbit

has a period $T \neq \tau$. Note that for small b_0 the subcritical orbit is unstable, while $z = 0$ is stable, but the respective exchanges of stability occur at slightly different values of b_0, corresponding to TC and subH. This is also corroborated by Fig. 2(a) (bottom), which displays the largest real part of the eigenvalues η of the steady state $z = 0$. The possible existence of such delay-induced periodic orbits with $T \neq \tau$, which results in a Floquet multiplier $\mu = 1$ of multiplicity two at TC, was overlooked in [20].

Next we analyse the conditions under which stabilization of the subcritical periodic orbit is possible. From Fig. 1(b) it is evident that the Pyragas curve must lie inside the yellow region, i.e., the Pyragas and Hopf curves emanating from the point $(\lambda, \tau) = (0, 2\pi)$ must locally satisfy the inequality $\tau_H(\lambda) < \tau_P(\lambda)$ for $\lambda < 0$. More generally, let us investigate the eigenvalue crossings of the Hopf eigenvalues $\eta = i\omega$ along the τ-axis of Fig. 1. In particular we derive conditions for the unstable dimensions of the trivial steady state near the Hopf bifurcation point $\lambda = 0$ in our model equation (2). On the τ-axis ($\lambda = 0$), the characteristic equation (6) for $\eta = i\omega$ is reduced to

$$\eta = i + b\left(e^{-\eta\tau} - 1\right), \tag{10}$$

and we obtain two series of Hopf points given by

$$0 \leq \tau_n^A \;=\; 2\pi n \tag{11}$$

$$0 < \tau_n^B \;=\; \frac{2\beta + 2\pi n}{1 - 2b_0 \sin\beta} \quad (n = 0, 1, 2, \dots). \tag{12}$$

The corresponding Hopf frequencies are $\omega^A = 1$ and $\omega^B = 1 - 2b_0 \sin\beta$, respectively. Note that series A consists of all Pyragas points, since $\tau_n^A = nT = \frac{2\pi n}{\omega^A}$. In the series B the integers n have to be chosen such that the delay $\tau_n^B \geq 0$. The case $b_0 \sin\beta = 1/2$, only, corresponds to $\omega^B = 0$ and does not occur for finite delays τ.

We evaluate the crossing directions of the critical Hopf eigenvalues next, along the positive τ-axis and for both series. Abbreviating $\frac{\partial}{\partial\tau}\eta$ by η_τ the crossing direction is given by sign(Re η_τ). Implicit differentiation of (10) with respect to τ at $\eta = i\omega$ implies

$$\text{sign}(\text{Re } \eta_\tau) = -\text{sign}(\omega)\,\text{sign}(\sin(\omega\tau - \beta)). \tag{13}$$

We are interested specifically in the Pyragas-Hopf points of series A (marked by red dots in Fig.1) where $\tau = \tau_n^A = 2\pi n$ and $\omega = \omega^A = 1$. Indeed sign(Re η_τ) = sign$(\sin\beta) > 0$ holds, provided we assume $0 < \beta < \pi$, *i.e.*, $b_I > 0$ for the feedback gain. This condition alone, however, is not sufficient to guarantee stability of the steady state for $\tau < 2n\pi$. We also have to consider the crossing direction sign(Re η_τ) along series B, $\omega^B = 1 - 2b_0 \sin\beta$, $\omega^B \tau_n^B = 2\beta + 2\pi n$, for $0 < \beta < \pi$. Eq. (13) now implies sign(Re η_τ) = sign$((2b_0 \sin\beta - 1)\sin\beta)$.

To compensate for the destabilization of $z = 0$ upon each crossing of any point $\tau_n^A = 2\pi n$, we must require stabilization (sign(Re η_τ) < 0) at each point τ_n^B of series B. This requires $0 < \beta < \arcsin(1/(2b_0))$ or $\pi - \arcsin(1/(2b_0)) < \beta < \pi$. The distance between two successive points τ_n^B and τ_{n+1}^B is $2\pi/\omega^B > 2\pi$. Therefore, there is at most one τ_n^B between any two successive Hopf points of series A. Stabilization requires exactly one such τ_n^B, specifically: $\tau_{k-1}^A < \tau_{k-1}^B < \tau_k^A$ for all $k = 1, 2, \dots$,n. This condition is satisfied if, and only if,

$$0 < \beta < \beta_n^*, \tag{14}$$

where $0 < \beta_n^* < \pi$ is the unique solution of the transcendental equation

$$\frac{1}{\pi}\beta_n^* + 2nb_0 \sin\beta_n^* = 1. \tag{15}$$

This holds because the condition $\tau_{k-1}^A < \tau_{k-1}^B < \tau_k^A$ first fails when $\tau_{k-1}^B = \tau_k^A$. Eq.(14) represents a necessary but not sufficient condition that the Pyragas choice $\tau_P = nT$ for the delay time will stabilize the periodic orbit.

To evaluate the second condition, $\tau_H < \tau_P$ near $(\lambda, \tau) = (0, 2\pi)$, we expand the exponential in the characteristic eq. (6) for $\omega\tau \approx 2\pi n$, and obtain the approximate Hopf curve for small $|\lambda|$:

$$\tau_H(\lambda) \approx 2\pi n - \frac{1}{b_I}(2\pi n b_R + 1)\lambda. \tag{16}$$

Recalling (5), the Pyragas stabilization condition $\tau_H(\lambda) < \tau_P(\lambda)$ is therefore satisfied for $\lambda < 0$ if, and only if,

$$\frac{1}{b_I}\left(b_R + \frac{1}{2\pi n}\right) < -\gamma. \tag{17}$$

Eq.(17) defines a domain in the plane of the complex feedback gain $b = b_R + ib_I = b_0 e^{i\beta}$ bounded from below (for $\gamma < 0 < b_I$) by the straight line

$$b_I = \frac{1}{-\gamma}\left(b_R + \frac{1}{2\pi n}\right). \tag{18}$$

Eq. (15) represents a curve $b_0(\beta)$, *i.e.*,

$$b_0 = \frac{1}{2n\sin\beta}\left(1 - \frac{\beta}{\pi}\right), \tag{19}$$

which forms the upper boundary of a domain given by the inequality (14). Thus (18) and (19) describe the boundaries of the domain of control in the complex plane of the feedback gain b in the limit of small λ. Fig.3 depicts this domain of control for $n = 1$, *i.e.*, a time-delay $\tau = \frac{2\pi}{1-\gamma\lambda}$. The lower and upper solid curves correspond to Eq. (18) and Eq. (19), respectively. The color code displays the numerical result of the largest real part, wherever < 0, of the Floquet exponent, calculated from linearization of the amplitude and phase equations around the periodic orbit. Outside the color shaded areas the periodic orbit is not stabilized. With increasing $|\lambda|$ the domain of stabilization shrinks, as the deviations from the linear

approximation (16) become larger. For sufficiently large $|\lambda|$ stabilization is no longer possible, in agreement with Fig.1(b). Note that for real values of b, i.e., $\beta = 0$, no stabilization occurs at all. Hence, stabilization fails if the feedback matrix B is a multiple of the identity matrix.

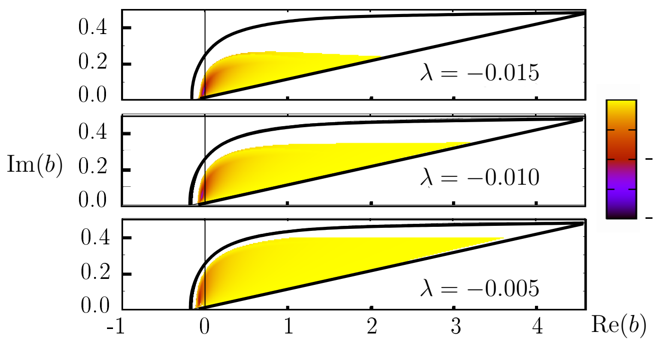

FIG. 3: (Color online) Domain of control in the plane of the complex feedback gain $b = b_0 e^{i\beta}$ for three different values of the bifurcation parameter λ. The black solid curves indicate the boundary of stability in the limit $\lambda \nearrow 0$, see (18), (19). The color-shading shows the magnitude of the largest (negative) real part of the Floquet exponents of the periodic orbit ($\gamma = -10$, $\tau = \frac{2\pi}{1-\gamma\lambda}$).

In conclusion, we have refuted a theorem which claims that a periodic orbit with an odd number of real Floquet multipliers greater than unity can never be stabilized by time-delayed feedback control. For this purpose we have analysed the generic example of the normal form of a subcritical Hopf bifurcation, which is paradigmatic for a large class of nonlinear systems. We have worked out explicit analytical conditions for stabilization of the periodic orbit generated by a subcritical Hopf bifurcation in terms of the amplitude and the phase of the feedback control gain [29]. Our results underline the crucial role of a non-vanishing phase of the control signal for stabilization of periodic orbits violating the odd number limitation. The feedback phase is readily accessible and can be adjusted, for instance, in laser systems, where subcritical Hopf bifurcation scenarios are abundant and Pyragas control can be realized via coupling to an external Fabry-Perot resonator [18]. The importance of the feedback phase for the stabilization of steady states in lasers [18] and neural systems [30], as well as for stabilization of periodic orbits by a time-delayed feedback control scheme using spatio-temporal filtering [31], has been noted recently. Here, we have shown that the odd number limitation does not hold in general, which opens up fundamental questions as well as a wide range of applications. The result will not only be important for practical applications in physical sciences, technology, and life sciences, where one might often desire to stabilize periodic orbits with an odd number of positive Floquet exponents, but also for tracking of unstable orbits and bifurcation analysis using time-delayed feedback control [32].

This work was supported by Deutsche Forschungsgemeinschaft in the framework of Sfb 555.

[1] H. G. Schuster (ed.), *Handbook of Chaos Control* (Wiley-VCH, Weinheim,1999).

[2] S. Boccaletti, C.Grebogi, Y. C. Lai, H. Mancini, and D. Maza, Phys. Rep. **329**, 103 (2000).

[3] D. J. Gauthier, Am. J. Phys. **71**, 750 (2003).

[4] E. Ott, C. Grebogi, and J. A. Yorke, Phys. Rev. Lett. **64**, 1196 (1990).

[5] K. Pyragas, Phys. Lett. A **170**, 421 (1992).

[6] K. Pyragas and A. Tamaševičius, Phys. Lett. A **180**, 99 (1993).

[7] S. Bielawski, D. Derozier, and P. Glorieux, Phys. Rev. E **49**, R971 (1994).

[8] T. Pierre, G. Bonhomme, and A. Atipo, Phys. Rev. Lett. **76**, 2290 (1996).

[9] K. Hall, D. J. Christini, M. Tremblay, J. J. Collins, L. Glass, and J. Billette, Phys. Rev. Lett. **78**, 4518 (1997).

[10] D. W. Sukow, M. E. Bleich, D. J. Gauthier, and J. E. S. Socolar, Chaos **7**, 560 (1997).

[11] O. Lüthje, S. Wolff, and G. Pfister, Phys. Rev. Lett. **86**, 1745 (2001).

[12] P. Parmananda, R. Madrigal, M. Rivera, L. Nyikos, I. Z. Kiss, and V. Gáspár, Phys. Rev. E **59**, 5266 (1999).

[13] J. M. Krodkiewski and J. S. Faragher, J. Sound Vib. **234**, 591 (2000).

[14] T. Fukuyama, H. Shirahama, and Y. Kawai, Phys. Plasmas **9**, 4525 (2002).

[15] C. von Loewenich, H. Benner, and W. Just, Phys. Rev. Lett. **93**, 174101 (2004).

[16] M. G. Rosenblum and A. S. Pikovsky, Phys. Rev. Lett. **92**, 114102 (2004).

[17] O. V. Popovych, C. Hauptmann, and P. A. Tass, Phys. Rev. Lett. **94**, 164102 (2005).

[18] S. Schikora, P. Hövel, H.-J. Wünsche, E. Schöll, and F. Henneberger, Phys. Rev. Lett. **97**, 213902 (2006).

[19] W. Just, T. Bernard, M. Ostheimer, E. Reibold, and H. Benner, Phys. Rev. Lett. **78**, 203 (1997).

[20] H. Nakajima, Phys. Lett. A **232**, 207 (1997).

[21] H. Nakajima and Y. Ueda, Physica D **111**, 143 (1998).

[22] I. Harrington and J. E. S. Socolar, Phys. Rev. E **64**, 056206 (2001).

[23] K. Pyragas, V. Pyragas, and H. Benner, Phys. Rev. E **70**, 056222 (2004).

[24] V. Pyragas and K. Pyragas, Phys. Rev. E **73**, 036215 (2006).

[25] J. E. S. Socolar, D. W. Sukow, and D. J. Gauthier, Phys. Rev. E **50**, 3245 (1994).

[26] H. G. Schuster and M. B. Stemmler, Phys. Rev. E **56**, 6410 (1997).

[27] K. Pyragas, Phys. Rev. Lett. **86**, 2265 (2001).

[28] O. Diekmann, S.A. van Gils, S.M. Verduyn Lunel, H.-O. Walther, *Delay Equations*, (Springer-Verlag, New York, 1995).

[29] For the complex conjugate values of b, stabilization of the periodic orbit can be shown by analogous arguments.

[30] M. G. Rosenblum and A. S. Pikovsky, Phys. Rev. E **70**, 041904 (2004).

[31] N. Baba, A. Amann, E. Schöll, and W. Just, Phys. Rev. Lett. **89**, 074101 (2002). W. Just, S. Popovich, A. Amann, N. Baba, and E. Schöll, Phys. Rev. E **67**,

026222 (2003).
[32] J. Sieber and B. Krauskopf, Appl. Nonlin. Math. Research Rep., U. of Bristol (2006), preprint.